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Two-component analogue of two-dimensional long wave–short wave resonance interaction equations: a derivation and solutions

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Abstract
The two-component analogue of two-dimensional long wave–short wave resonance interaction equations is derived in a physical setting. Wronskian solutions of the integrable two-component analogue of two-dimensional long wave–short wave resonance interaction equations are presented.

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1. Introduction

Recently, vector soliton equations (or coupled soliton equations) such as the vector nonlinear Schrödinger (vNLS) equation have received much attention in mathematical physics and nonlinear physics. Especially, the vNLS equation has been studied by several researchers from both mathematical and physical points of view [1–5]. It was pointed out that vector solitons can be used in the construction of logic gates [6–8]. It was also pointed out that the Yang–Baxter map is the key to understanding the mathematical structure of logic gates based on vector solitons [4, 9, 10].

Although there are many works about one-dimensional vector solitons, a mathematical work about two-dimensional vector solitons is still missing. For a more complete understanding of the mathematical structure of vector solitons, the study of two-dimensional vector solitons is very important.

In this paper, we derive a two-component analogue of two-dimensional long wave–short wave resonance interaction (2c–2d-LSRI) equations in a physical setting. We also present the Wronskian solution of the integrable 2c–2d-LSRI equations.

2. Derivation

Consider the interaction of nonlinear dispersive waves on three channels, e.g. laser beams on some dispersive material. Suppose that the dispersion relations of these weakly nonlinear waves are

$$\omega_i = \omega_i(k_{x,i}, k_{y,i} : |A_1|^2, |A_2|^2, |A_3|^2), \qquad \text{for} \quad i = 1, 2, 3,$$

where ω_i and A_i are angular frequencies and amplitudes of each channel i, respectively. Suppose that carrier wave is expressed by $\exp(\mathrm{i}(k_{x,0}x + k_{y,0}y - \omega_0 t))$. Taylor expansion around $\mathbf{k}_0 = (k_{x,0}, k_{y,0})$, ω_0 and $|A_i| = 0$ makes

$$\begin{aligned}\omega_i - \omega_0 = {}& \left(\frac{\partial\omega_i}{\partial k_{x,i}}\right)_0 (k_{x,i} - k_{x,0}) + \left(\frac{\partial\omega_i}{\partial k_{y,i}}\right)_0 (k_{y,i} - k_{y,0}) + \frac{1}{2}\left(\frac{\partial^2\omega_i}{\partial k_{x,i}^2}\right)_0 (k_{x,i} - k_{x,0})^2 \\ & + \frac{1}{2}\left(\frac{\partial^2\omega_i}{\partial k_{y,i}^2}\right)_0 (k_{y,i} - k_{y,0})^2 + \left(\frac{\partial^2\omega_i}{\partial k_{x,i} k_{y,i}}\right)_0 (k_{x,i} - k_{x,0})(k_{y,i} - k_{y,0}) \\ & + \left(\frac{\partial\omega_i}{\partial|A_1|^2}\right)_0 |A_1|^2 + \left(\frac{\partial\omega_i}{\partial|A_2|^2}\right)_0 |A_2|^2 + \left(\frac{\partial\omega_i}{\partial|A_3|^2}\right)_0 |A_3|^2 + \cdots, \\ & \text{for} \quad i = 1, 2, 3, \end{aligned} \tag{2.1}$$

where the subscript 0 of $(\ \ )_0$ means setting $k_{x,i} = k_{x,0}, k_{y,i} = k_{y,0}, \omega_i = \omega_0$ and $|A_i| = 0$. Replacing $\omega_i, k_{x,i}$ and $k_{y,i}$ to operators by the rules $\omega_i - \omega_0 \sim \mathrm{i}\partial/\partial t, k_{x,i} - k_{x,0} \sim -\mathrm{i}\partial/\partial x, k_{y,i} - k_{y,0} \sim -\mathrm{i}\partial/\partial y$, and applying those equations to $A_i(x, y, t)$, we obtain

$$\begin{aligned}\mathrm{i}\frac{\partial A_i}{\partial t} + \mathrm{i}\left(\frac{\partial\omega_i}{\partial k_{x,i}}\right)_0 \frac{\partial A_i}{\partial x} + \mathrm{i}\left(\frac{\partial\omega_i}{\partial k_{y,i}}\right)_0 \frac{\partial A_i}{\partial y} + \frac{1}{2}\left(\frac{\partial^2\omega_i}{\partial k_{x,i}^2}\right)_0 \frac{\partial^2 A_i}{\partial x^2} + \frac{1}{2}\left(\frac{\partial^2\omega_i}{\partial k_{y,i}^2}\right)_0 \frac{\partial^2 A_i}{\partial y^2} \\ + \left(\frac{\partial^2\omega_i}{\partial k_{x,i}\partial k_{y,i}}\right)_0 \frac{\partial^2 A_i}{\partial x\partial y} - \left(\frac{\partial\omega_i}{\partial|A_1|^2}\right)_0 |A_1|^2 A_i - \left(\frac{\partial\omega_i}{\partial|A_2|^2}\right)_0 |A_2|^2 A_i \\ - \left(\frac{\partial\omega_i}{\partial|A_3|^2}\right)_0 |A_3|^2 A_i = 0, \qquad \text{for} \quad i = 1, 2, 3. \end{aligned} \tag{2.2}$$

By the transformation of coordinates

$$x' = x - \left(\frac{\partial\omega_3}{\partial k_{x,3}}\right)_0 t, \qquad y' = y - \left(\frac{\partial\omega_3}{\partial k_{y,3}}\right)_0 t, \qquad t' = t,$$

we obtain

$$\begin{aligned}\mathrm{i}\frac{\partial A_1}{\partial t} + \mathrm{i}v_{x,1}\frac{\partial A_1}{\partial x} + \mathrm{i}v_{y,1}\frac{\partial A_1}{\partial y} + \frac{1}{2}\left(\frac{\partial^2\omega_1}{\partial k_{x,1}^2}\right)_0 \frac{\partial^2 A_1}{\partial x^2} + \frac{1}{2}\left(\frac{\partial^2\omega_1}{\partial k_{y,1}^2}\right)_0 \frac{\partial^2 A_1}{\partial y^2} + \left(\frac{\partial^2\omega_1}{\partial k_{x,1}\partial k_{y,1}}\right)_0 \frac{\partial^2 A_1}{\partial x\partial y} \\ - \left(\frac{\partial\omega_1}{\partial|A_1|^2}\right)_0 |A_1|^2 A_1 - \left(\frac{\partial\omega_1}{\partial|A_2|^2}\right)_0 |A_2|^2 A_1 - \left(\frac{\partial\omega_1}{\partial|A_3|^2}\right)_0 |A_3|^2 A_1 = 0,\end{aligned} \tag{2.3}$$

$$\begin{aligned}\mathrm{i}\frac{\partial A_2}{\partial t} + \mathrm{i}v_{x,2}\frac{\partial A_2}{\partial x} + \mathrm{i}v_{y,2}\frac{\partial A_2}{\partial y} + \frac{1}{2}\left(\frac{\partial^2\omega_2}{\partial k_{x,2}^2}\right)_0 \frac{\partial^2 A_2}{\partial x^2} + \frac{1}{2}\left(\frac{\partial^2\omega_2}{\partial k_{y,2}^2}\right)_0 \frac{\partial^2 A_2}{\partial y^2} + \left(\frac{\partial^2\omega_2}{\partial k_{x,2}\partial k_{y,2}}\right)_0 \frac{\partial^2 A_2}{\partial x\partial y} \\ - \left(\frac{\partial\omega_2}{\partial|A_1|^2}\right)_0 |A_1|^2 A_2 - \left(\frac{\partial\omega_2}{\partial|A_2|^2}\right)_0 |A_2|^2 A_2 - \left(\frac{\partial\omega_2}{\partial|A_3|^2}\right)_0 |A_3|^2 A_2 = 0,\end{aligned} \tag{2.4}$$

$$\mathrm{i}\frac{\partial A_3}{\partial t}+\frac{1}{2}\left(\frac{\partial^2\omega_3}{\partial k_{x,3}^2}\right)_0\frac{\partial^2 A_3}{\partial x^2}+\frac{1}{2}\left(\frac{\partial^2\omega_3}{\partial k_{y,3}^2}\right)_0\frac{\partial^2 A_3}{\partial y^2}+\left(\frac{\partial^2\omega_3}{\partial k_{x,3}\partial k_{y,3}}\right)_0\frac{\partial^2 A_3}{\partial x\partial y}-\left(\frac{\partial\omega_3}{\partial|A_1|^2}\right)_0|A_1|^2A_3 -\left(\frac{\partial\omega_3}{\partial|A_2|^2}\right)_0|A_2|^2A_3-\left(\frac{\partial\omega_3}{\partial|A_3|^2}\right)_0|A_3|^2A_3=0, \tag{2.5}$$

where

$$v_{x,1}=\left(\frac{\partial\omega_1}{\partial k_{x,1}}\right)_0-\left(\frac{\partial\omega_3}{\partial k_{x,3}}\right)_0,\qquad v_{y,1}=\left(\frac{\partial\omega_1}{\partial k_{y,1}}\right)_0-\left(\frac{\partial\omega_3}{\partial k_{y,3}}\right)_0,$$

$$v_{x,2}=\left(\frac{\partial\omega_2}{\partial k_{x,2}}\right)_0-\left(\frac{\partial\omega_3}{\partial k_{x,3}}\right)_0,\qquad v_{y,2}=\left(\frac{\partial\omega_2}{\partial k_{y,2}}\right)_0-\left(\frac{\partial\omega_3}{\partial k_{y,3}}\right)_0,$$

and x', y', t' are replaced by x, y, t. This is a generalization of the vector nonlinear Schrödinger equation (Manakov system) [2, 11].

Now, consider the situation in which $v_{x,1}$ and $v_{x,2}$ are neglected. For notational convenience, we rewrite the above equations as

$$\mathrm{i}\frac{\partial A_1}{\partial t}+\mathrm{i}v_{y,1}\frac{\partial A_1}{\partial y}+\alpha_1\frac{\partial^2 A_1}{\partial x^2}+\alpha_2\frac{\partial^2 A_1}{\partial y^2}+\alpha_3\frac{\partial^2 A_1}{\partial x\partial y}+\alpha_4|A_1|^2A_1+\alpha_5|A_2|^2A_1+\alpha_6|A_3|^2A_1=0, \tag{2.6}$$

$$\mathrm{i}\frac{\partial A_2}{\partial t}+\mathrm{i}v_{y,2}\frac{\partial A_2}{\partial y}+\beta_1\frac{\partial^2 A_2}{\partial x^2}+\beta_2\frac{\partial^2 A_2}{\partial y^2}+\beta_3\frac{\partial^2 A_2}{\partial x\partial y}+\beta_4|A_1|^2A_2+\beta_5|A_2|^2A_2+\beta_6|A_3|^2A_2=0, \tag{2.7}$$

$$\mathrm{i}\frac{\partial A_3}{\partial t}+\gamma_1\frac{\partial^2 A_3}{\partial x^2}+\gamma_2\frac{\partial^2 A_3}{\partial y^2}+\gamma_3\frac{\partial^2 A_3}{\partial x\partial y}+\gamma_4|A_1|^2A_3+\gamma_5|A_2|^2A_3+\gamma_6|A_3|^2A_3=0. \tag{2.8}$$

Assume that the channel 3 is normal dispersion and the channels 1 and 2 are anomalous dispersion. We study the dark pulses generated in the channel 3:[12]

$$A_1=\psi_1\exp(\mathrm{i}\delta_1 t),\quad A_2=\psi_2\exp(\mathrm{i}\delta_2 t),\quad A_3=(u_0+a(x,y,t))\exp(\mathrm{i}\Gamma t+\mathrm{i}\phi(x,y,t)),$$

$$\delta_1=-\left(\frac{\partial\omega_1}{\partial|A_3|^2}\right)_0u_0^2,\qquad \delta_2=-\left(\frac{\partial\omega_2}{\partial|A_3|^2}\right)_0u_0^2,\qquad \Gamma=-\left(\frac{\partial\omega_3}{\partial|A_3|^2}\right)_0u_0^2,$$

where a and $\psi_i (i=1,2)$ are small. Substituting these into equations (2.6)–(2.8), we obtain

$$\frac{\partial a}{\partial t}+\gamma_1u_0\frac{\partial^2\phi}{\partial x^2}+\gamma_2u_0\frac{\partial^2\phi}{\partial y^2}+\gamma_3u_0\frac{\partial^2\phi}{\partial x\partial y}=0, \tag{2.9}$$

$$-u_0\frac{\partial\phi}{\partial t}+\gamma_1\frac{\partial^2 a}{\partial x^2}+\gamma_2\frac{\partial^2 a}{\partial y^2}+\gamma_3\frac{\partial^2 a}{\partial x\partial y}+\gamma_4u_0|\psi_1|^2+\gamma_5u_0|\psi_2|^2+3\gamma_6u_0^2a=0, \tag{2.10}$$

$$\mathrm{i}\frac{\partial\psi_1}{\partial t}+\mathrm{i}v_{y,1}\frac{\partial\psi_1}{\partial y}+\alpha_1\frac{\partial^2\psi_1}{\partial x^2}+\alpha_2\frac{\partial^2\psi_1}{\partial y^2}+\alpha_3\frac{\partial^2\psi_1}{\partial x\partial y}+\alpha_4|\psi_1|^2\psi_1+\alpha_5|\psi_2|^2\psi_1+2\alpha_6u_0a\psi_1=0, \tag{2.11}$$

$$\mathrm{i}\frac{\partial\psi_2}{\partial t}+\mathrm{i}v_{y,2}\frac{\partial\psi_2}{\partial y}+\beta_1\frac{\partial^2\psi_2}{\partial x^2}+\beta_2\frac{\partial^2\psi_2}{\partial y^2}+\beta_3\frac{\partial^2\psi_2}{\partial x\partial y}+\beta_4|\psi_1|^2\psi_2+\beta_5|\psi_2|^2\psi_2+2\beta_6u_0a\psi_2=0. \tag{2.12}$$

Assume that the y-dependency of ϕ can be neglected, i.e. we can neglect ϕ_y and ϕ_{yy}. Then equation (2.9) reduces to

$$\frac{\partial a}{\partial t} + \gamma_1 u_0 \frac{\partial^2 \phi}{\partial x^2} = 0,$$

i.e.,

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{1}{\gamma_1 u_0} \frac{\partial a}{\partial t}.$$

Substituting this into equation (2.10), we have

$$\frac{\partial^2 a}{\partial t^2} + 3\gamma_1\gamma_6 u_0^2 \frac{\partial^2 a}{\partial x^2} + \gamma_1^2 \frac{\partial^4 a}{\partial x^4} + \gamma_1\gamma_2 \frac{\partial^4 a}{\partial x^2 \partial y^2} + \gamma_1\gamma_3 \frac{\partial^4 a}{\partial x^3 \partial y} + \gamma_1 \frac{\partial^2}{\partial x^2}(\gamma_4 u_0 |\psi_1|^2 + \gamma_5 u_0 |\psi_2|^2) = 0. \tag{2.13}$$

By

$$t' = \varepsilon t, \qquad x' = \varepsilon^{1/2}(x + ct), \qquad y' = \varepsilon y,$$

$\left(c = 3\gamma_1\gamma_6 u_0^2\right)$ with $a = \varepsilon a_0$, $\psi_1 = \varepsilon^{3/4}\Phi_1$, $\psi_2 = \varepsilon^{3/4}\Phi_2$ (ε is small), we obtain equations of lowest order of ε:

$$2c \frac{\partial^2 a}{\partial x \partial t} + \gamma_1 \frac{\partial^2}{\partial x^2}(\gamma_4 u_0 |\psi_1|^2 + \gamma_5 u_0 |\psi_2|^2) = 0, \tag{2.14}$$

$$\mathrm{i}\frac{\partial \psi_1}{\partial t} + \mathrm{i} v_{y,1} \frac{\partial \psi_1}{\partial y} + \alpha_1 \frac{\partial^2 \psi_1}{\partial x^2} + 2\alpha_6 u_0 a \psi_1 = 0, \tag{2.15}$$

$$\mathrm{i}\frac{\partial \psi_2}{\partial t} + \mathrm{i} v_{y,2} \frac{\partial \psi_2}{\partial y} + \beta_1 \frac{\partial^2 \psi_2}{\partial x^2} + 2\beta_6 u_0 a \psi_2 = 0. \tag{2.16}$$

Here we again have disregarded the primes and have replaced a_0, Φ_1 and Φ_2 with a, ψ_1 and ψ_2. The first equation leads to

$$2c \frac{\partial a}{\partial t} + \gamma_1 \frac{\partial}{\partial x}(\gamma_4 u_0 |\psi_1|^2 + \gamma_5 u_0 |\psi_2|^2) = 0. \tag{2.17}$$

This system is nothing but the two-component analogue of two-dimensional analogue of the long wave–short wave resonance interaction (2c–2d-LSRI) equations[13–16]. Note that the special case of coefficients ($v_{y,1} = v_{y,2}$, $\alpha_1 = \beta_1$, $\alpha_6 = \beta_6$, $\gamma_4 = \gamma_5$) is integrable. In the one-component integrable case, several solutions have been presented [15, 17–19]. In [20], some solutions of matrix generalization were discussed. In the following section, we will present the determinant formula of N-soliton solution for the integrable 2c–2d-LSRI equations.

3. Soliton solutions

We study the soliton solutions of an integrable case of the 2c–2d-LSRI equations

$$\mathrm{i}\left(S_t^{(1)} + S_y^{(1)}\right) - S_{xx}^{(1)} + L S^{(1)} = 0, \tag{3.1}$$

$$\mathrm{i}\left(S_t^{(2)} + S_y^{(2)}\right) - S_{xx}^{(2)} + L S^{(2)} = 0, \tag{3.2}$$

$$L_t = 2(|S^{(1)}|^2)_x + 2(|S^{(2)}|^2)_x. \tag{3.3}$$

By the dependent variable transformation

$$S^{(1)} = \frac{G}{F}, \qquad S^{(2)} = \frac{H}{F}, \qquad L = -2\frac{\partial^2}{\partial x^2} \log F, \tag{3.4}$$

we have three bilinear equations

$$\left[\mathrm{i}(D_t + D_y) - D_x^2\right] G \cdot F = 0, \tag{3.5}$$

$$\left[\mathrm{i}(D_t + D_y) - D_x^2\right] H \cdot F = 0, \tag{3.6}$$

$$(D_t D_x - 2c) F \cdot F + 2GG^* + 2HH^* = 0. \tag{3.7}$$

Here we set $c = 0$ which means we consider the bright-type soliton solutions. These bilinear equations belong to the three-component KP hierarchy [21–24]. We can construct soliton solutions in which the number of solitons in channel i is N_i $(i = 1, 2, 3)$. We call this solution (N_1, N_2, N_3)-soliton solution.

Wronskian form of the bright type $(N, M, N + M)$*-soliton solutions*:

Let

$$\tau_{nm}^{NM} = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \cdots & \varphi_1^{(N+M-1+n+m)} & \psi_1 & \psi_1^{(1)} & \cdots & \psi_1^{(N-1-n)} & 0 & 0 & \cdots & 0 \\ \varphi_2 & \varphi_2^{(1)} & \cdots & \varphi_2^{(N+M-1+n+m)} & \psi_2 & \psi_2^{(1)} & \cdots & \psi_2^{(N-1-n)} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \varphi_{2N} & \varphi_{2N}^{(1)} & \cdots & \varphi_{2N}^{(N+M-1+n+m)} & \psi_{2N} & \psi_{2N}^{(1)} & \cdots & \psi_{2N}^{(N-1-n)} & 0 & 0 & \cdots & 0 \\ \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(N+M-1+n+m)} & 0 & 0 & \cdots & 0 & \chi_1 & \chi_1^{(1)} & \cdots & \chi_1^{(M-1-m)} \\ \phi_2 & \phi_2^{(1)} & \cdots & \phi_2^{(N+M-1+n+m)} & 0 & 0 & \cdots & 0 & \chi_2 & \chi_2^{(1)} & \cdots & \chi_2^{(M-1-m)} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \phi_{2M} & \phi_{2M}^{(1)} & \cdots & \phi_{2M}^{(N+M-1+n+m)} & 0 & 0 & \cdots & 0 & \chi_{2M} & \chi_{2M}^{(1)} & \cdots & \chi_{2M}^{(M-1-m)} \end{vmatrix}$$

$$\begin{aligned}
&\varphi_i = \mathrm{e}^{\xi_i}, && \xi_i = p_i x_1 + p_i^2 x_2, && \text{for} \quad i = 1, 2, \cdots, N \\
&\varphi_{N+i} = \mathrm{e}^{-\xi_i^*}, && -\xi_i^* = -p_i^* x_1 + (-p_i^*)^2 x_2, && \text{for} \quad i = 1, 2, \cdots, N \\
&\phi_i = \mathrm{e}^{\theta_i}, && \theta_i = s_i x_1 + s_i^2 x_2, && \text{for} \quad i = 1, 2, \cdots, M \\
&\phi_{M+i} = \mathrm{e}^{-\theta_i^*}, && -\theta_i^* = -s_i^* x_1 + (-s_i^*)^2 x_2, && \text{for} \quad i = 1, 2, \cdots, M \\
&\psi_i = a_i\, \mathrm{e}^{\eta_i}, && \eta_i = q_i y_1 + \eta_{i0}, && \text{for} \quad i = 1, 2, \cdots, N \\
&\psi_{N+i} = a_{N+i}\, \mathrm{e}^{-\eta_i^*}, && -\eta_i^* = -q_i^* y_1 - \eta_{i0}^*, && \text{for} \quad i = 1, 2, \cdots, N \\
&\chi_i = b_i\, \mathrm{e}^{\zeta_i}, && \zeta_i = r_i z_1 + \zeta_{i0}, && \text{for} \quad i = 1, 2, \cdots, M \\
&\chi_{M+i} = b_{M+i}\, \mathrm{e}^{-\zeta_i^*}, && -\zeta_i^* = -r_i^* z_1 - \zeta_{i0}^*, && \text{for} \quad i = 1, 2, \cdots, M
\end{aligned}$$

$$a_i = \left(\prod_{\substack{k=1\\k\neq i}}^{N} \frac{p_k - p_i}{q_k - q_i}\right)\left(\prod_{l=1}^{M} (s_l - p_i)\right), \qquad \text{for} \quad i = 1, 2, \cdots, N$$

$$a_{N+i} = \varepsilon_i \left(\prod_{k=1}^{N} \frac{p_k + p_i^*}{q_k + q_i^*}\right)\left(\prod_{l=1}^{M} (s_l + p_i^*)\right), \qquad \text{for} \quad i = 1, 2, \cdots, N$$

$$b_i = \left(\prod_{k=1}^{N} (p_k - s_i)\right)\left(\prod_{\substack{l=1\\l\neq i}}^{M} \frac{s_l - s_i}{r_l - r_i}\right), \qquad \text{for} \quad i = 1, 2, \cdots, M$$

$$b_{M+i} = \delta_i \left(\prod_{k=1}^{N} (p_k + s_i^*)\right)\left(\prod_{l=1}^{M} \frac{s_l + s_i^*}{r_l + r_i^*}\right), \qquad \text{for} \quad i = 1, 2, \cdots, M$$

$$\varepsilon_i = \pm 1, \qquad \delta_i = \pm 1,$$

where * means complex conjugate and p_i, q_i $(1 \leqslant i \leqslant N)$ and s_i, r_i $(1 \leqslant i \leqslant M)$ are complex wave numbers, and η_{i0} $(1 \leqslant i \leqslant N)$ and ζ_{i0} $(1 \leqslant i \leqslant M)$ are complex phase parameters. In order to obtain regular solutions, we have to choose appropriate sign for ε_i and δ_i, which depend on parameters p_i, q_i, r_i, s_i. We take

$$x_1 = x, \qquad x_2 = -\mathrm{i}y, \qquad y_1 = y - t, \qquad z_1 = y - t,$$

where x, y and t are real , (i.e. x_1, y_1 and z_1 are real and x_2 is pure imaginary).

Let

$$f = \tau_{00}, \qquad g = \tau_{10}, \qquad \bar{g} = \tau_{-1,0}, \qquad h = \tau_{01}, \qquad \bar{h} = \tau_{0,-1}.$$

These tau-functions satisfy the condition

$$\left(\frac{g}{f}\right)^* = \frac{\bar{g}}{f}, \qquad \left(\frac{h}{f}\right)^* = \frac{\bar{h}}{f},$$

$$f\mathcal{G} : \text{real},$$

where $\mathcal{G}$ is an exponential factor which is a gauge function (see the appendix). Let $F = f\mathcal{G}$, $G = g\mathcal{G}$, $G^* = \bar{g}\mathcal{G}$, $H = h\mathcal{G}$, $H^* = \bar{h}\mathcal{G}$. The functions F, G and H satisfy the bilinear equations (3.5)–(3.7) and reality of F and complex conjugacy of G and H. The function $L = -2\frac{\partial^2}{\partial x^2}\log F$ represents $(N+M)$-soliton solution, $S_1 = G/F$ represents N-soliton solution and $S_2 = H/F$ represents M-soliton solution.

(1, 1, 2)-*soliton solution*

The τ-functions of (1, 1, 2)-soliton solution are the following:

$$f = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \psi_1 & 0 \\ \varphi_2 & \varphi_2^{(1)} & \psi_2 & 0 \\ \phi_1 & \phi_1^{(1)} & 0 & \chi_1 \\ \phi_2 & \phi_2^{(1)} & 0 & \chi_2 \end{vmatrix}, \qquad g = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \varphi_1^{(2)} & 0 \\ \varphi_2 & \varphi_2^{(1)} & \varphi_2^{(2)} & 0 \\ \phi_1 & \phi_1^{(1)} & \phi_1^{(2)} & \chi_1 \\ \phi_2 & \phi_2^{(1)} & \phi_2^{(2)} & \chi_2 \end{vmatrix}, \qquad \bar{g} = \begin{vmatrix} \varphi_1 & \psi_1 & \psi_1^{(1)} & 0 \\ \varphi_2 & \psi_2 & \psi_2^{(1)} & 0 \\ \phi_1 & 0 & 0 & \chi_1 \\ \phi_2 & 0 & 0 & \chi_2 \end{vmatrix},$$

$$h = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \varphi_1^{(2)} & \psi_1 \\ \varphi_2 & \varphi_2^{(1)} & \varphi_2^{(2)} & \psi_2 \\ \phi_1 & \phi_1^{(1)} & \phi_1^{(2)} & 0 \\ \phi_2 & \phi_2^{(1)} & \phi_2^{(2)} & 0 \end{vmatrix}, \qquad \bar{h} = \begin{vmatrix} \varphi_1 & \psi_1 & 0 & 0 \\ \varphi_2 & \psi_2 & 0 & 0 \\ \phi_1 & 0 & \chi_1 & \chi_1^{(1)} \\ \phi_2 & 0 & \chi_2 & \chi_2^{(1)} \end{vmatrix},$$

$$\begin{aligned}
&\varphi_1 = \mathrm{e}^{\xi_1}, && \xi_1 = p_1x_1 + p_1^2x_2, && \varphi_2 = \mathrm{e}^{-\xi_1^*}, && -\xi_1^* = -p_1^*x_1 + (-p_1^*)^2x_2, \\
&\phi_1 = \mathrm{e}^{\theta_1}, && \theta_1 = s_1x_1 + s_1^2x_2, && \phi_2 = \mathrm{e}^{-\theta_1^*}, && -\theta_1^* = -s_1^*x_1 + (-s_1^*)^2x_2, \\
&\psi_1 = a_1\,\mathrm{e}^{\eta_1}, && \eta_1 = q_1y_1 + \eta_{10}, && \psi_2 = a_2\,\mathrm{e}^{-\eta_1^*}, && -\eta_1^* = -q_1^*y_1 - \eta_{10}^*, \\
&\chi_1 = b_1\,\mathrm{e}^{\zeta_1}, && \zeta_1 = r_1z_1 + \zeta_{10}, && \chi_2 = b_2\,\mathrm{e}^{-\zeta_1^*}, && -\zeta_1^* = -r_1^*z_1 - \zeta_{10}^*,
\end{aligned}$$

$$a_1 = s_1 - p_1, \qquad b_1 = p_1 - s_1, \qquad a_2 = \varepsilon(s_1 + p_1^*)\frac{p_1 + p_1^*}{q_1 + q_1^*}, \qquad b_2 = \delta(s_1^* + p_1)\frac{s_1 + s_1^*}{r_1 + r_1^*},$$

$$\varepsilon = \pm 1, \qquad \delta = \pm 1,$$

and $x_1 = x$, $x_2 = -\mathrm{i}y$, $y_1 = y - t$, $z_1 = y - t$.

Figure 1 shows the (1, 1, 2)-soliton solution. In the fields S_1 and S_2, there is the single soliton. However, each solitons in S_1 and S_2 interact through the field L. So the behaviour of solitons looks like two-soliton solution, i.e., we see the phase shift in the region of the interaction (see the bottom graph in figure 1).

(2, 2, 4)-*soliton solution*

The τ-functions of (2, 2, 4)-soliton solution are the following:

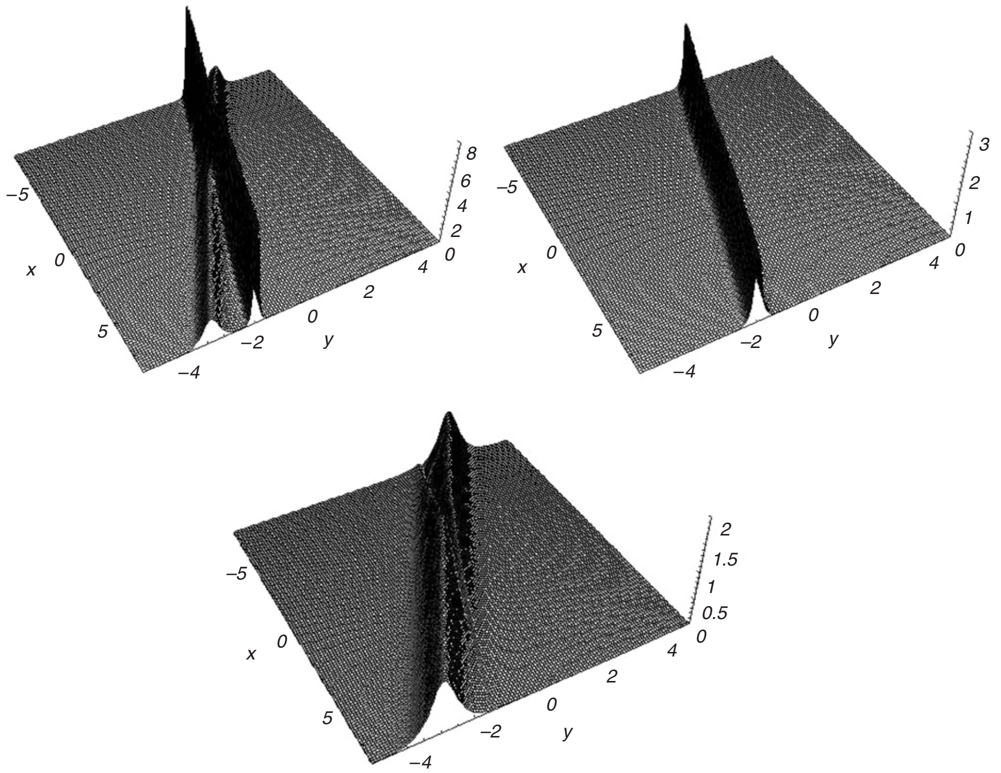

Figure 1. (1, 1, 2)-soliton solution. $p_1 = 2 + 2\mathrm{i}, s_1 = -1 + \mathrm{i}, q_1 = -2 + \mathrm{i}, r_1 = 1 + \mathrm{i}, \varepsilon = \delta = 1$. The top left graph is $-L$, the top right graph is $S^{(1)}$ and the bottom graph is $S^{(2)}$ at $t = 0$.

$$f = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \varphi_1^{(2)} & \varphi_1^{(3)} & \psi_1 & \psi_1^{(1)} & 0 & 0 \\ \varphi_2 & \varphi_2^{(1)} & \varphi_2^{(2)} & \varphi_2^{(3)} & \psi_2 & \psi_2^{(1)} & 0 & 0 \\ \varphi_3 & \varphi_3^{(1)} & \varphi_3^{(2)} & \varphi_3^{(3)} & \psi_3 & \psi_3^{(1)} & 0 & 0 \\ \varphi_4 & \varphi_4^{(1)} & \varphi_4^{(2)} & \varphi_4^{(3)} & \psi_4 & \psi_4^{(1)} & 0 & 0 \\ \phi_1 & \phi_1^{(1)} & \phi_1^{(2)} & \phi_1^{(3)} & 0 & 0 & \chi_1 & \chi_1^{(1)} \\ \phi_2 & \phi_2^{(1)} & \phi_2^{(2)} & \phi_2^{(3)} & 0 & 0 & \chi_2 & \chi_2^{(1)} \\ \phi_3 & \phi_3^{(1)} & \phi_3^{(2)} & \phi_3^{(3)} & 0 & 0 & \chi_3 & \chi_3^{(1)} \\ \phi_4 & \phi_4^{(1)} & \phi_4^{(2)} & \phi_4^{(3)} & 0 & 0 & \chi_4 & \chi_4^{(1)} \end{vmatrix}$$

$$g = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \varphi_1^{(2)} & \varphi_1^{(3)} & \varphi_1^{(4)} & \psi_1 & 0 & 0 \\ \varphi_2 & \varphi_2^{(1)} & \varphi_2^{(2)} & \varphi_2^{(3)} & \varphi_2^{(4)} & \psi_2 & 0 & 0 \\ \varphi_3 & \varphi_3^{(1)} & \varphi_3^{(2)} & \varphi_3^{(3)} & \varphi_3^{(4)} & \psi_3 & 0 & 0 \\ \varphi_4 & \varphi_4^{(1)} & \varphi_4^{(2)} & \varphi_4^{(3)} & \varphi_4^{(4)} & \psi_4 & 0 & 0 \\ \phi_1 & \phi_1^{(1)} & \phi_1^{(2)} & \phi_1^{(3)} & \phi_1^{(4)} & 0 & \chi_1 & \chi_1^{(1)} \\ \phi_2 & \phi_2^{(1)} & \phi_2^{(2)} & \phi_2^{(3)} & \phi_2^{(4)} & 0 & \chi_2 & \chi_2^{(1)} \\ \phi_3 & \phi_3^{(1)} & \phi_3^{(2)} & \phi_3^{(3)} & \phi_3^{(4)} & 0 & \chi_3 & \chi_3^{(1)} \\ \phi_4 & \phi_4^{(1)} & \phi_4^{(2)} & \phi_4^{(3)} & \phi_4^{(4)} & 0 & \chi_4 & \chi_4^{(1)} \end{vmatrix}$$

$$\bar{g} = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \varphi_1^{(2)} & \psi_1 & \psi_1^{(1)} & \psi_1^{(2)} & 0 & 0 \\ \varphi_2 & \varphi_2^{(1)} & \varphi_2^{(2)} & \psi_2 & \psi_2^{(1)} & \psi_2^{(2)} & 0 & 0 \\ \varphi_3 & \varphi_3^{(1)} & \varphi_3^{(2)} & \psi_3 & \psi_3^{(1)} & \psi_3^{(2)} & 0 & 0 \\ \varphi_4 & \varphi_4^{(1)} & \varphi_4^{(2)} & \psi_4 & \psi_4^{(1)} & \psi_4^{(2)} & 0 & 0 \\ \phi_1 & \phi_1^{(1)} & \phi_1^{(2)} & 0 & 0 & 0 & \chi_1 & \chi_1^{(1)} \\ \phi_2 & \phi_2^{(1)} & \phi_2^{(2)} & 0 & 0 & 0 & \chi_2 & \chi_2^{(1)} \\ \phi_3 & \phi_3^{(1)} & \phi_3^{(2)} & 0 & 0 & 0 & \chi_3 & \chi_3^{(1)} \\ \phi_4 & \phi_4^{(1)} & \phi_4^{(2)} & 0 & 0 & 0 & \chi_4 & \chi_4^{(1)} \end{vmatrix}$$

$$h = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \varphi_1^{(2)} & \varphi_1^{(3)} & \varphi_1^{(4)} & \psi_1 & \psi_1^{(1)} & 0 \\ \varphi_2 & \varphi_2^{(1)} & \varphi_2^{(2)} & \varphi_2^{(3)} & \varphi_2^{(4)} & \psi_2 & \psi_2^{(1)} & 0 \\ \varphi_3 & \varphi_3^{(1)} & \varphi_3^{(2)} & \varphi_3^{(3)} & \varphi_3^{(4)} & \psi_3 & \psi_3^{(1)} & 0 \\ \varphi_4 & \varphi_4^{(1)} & \varphi_4^{(2)} & \varphi_4^{(3)} & \varphi_4^{(4)} & \psi_4 & \psi_4^{(1)} & 0 \\ \phi_1 & \phi_1^{(1)} & \phi_1^{(2)} & \phi_1^{(3)} & \phi_1^{(4)} & 0 & 0 & \chi_1 \\ \phi_2 & \phi_2^{(1)} & \phi_2^{(2)} & \phi_2^{(3)} & \phi_2^{(4)} & 0 & 0 & \chi_2 \\ \phi_3 & \phi_3^{(1)} & \phi_3^{(2)} & \phi_3^{(3)} & \phi_3^{(4)} & 0 & 0 & \chi_3 \\ \phi_4 & \phi_4^{(1)} & \phi_4^{(2)} & \phi_4^{(3)} & \phi_4^{(4)} & 0 & 0 & \chi_4 \end{vmatrix}$$

$$\bar{h} = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \varphi_1^{(2)} & \psi_1 & \psi_1^{(1)} & 0 & 0 & 0 \\ \varphi_2 & \varphi_2^{(1)} & \varphi_2^{(2)} & \psi_2 & \psi_2^{(1)} & 0 & 0 & 0 \\ \varphi_3 & \varphi_3^{(1)} & \varphi_3^{(2)} & \psi_3 & \psi_3^{(1)} & 0 & 0 & 0 \\ \varphi_4 & \varphi_4^{(1)} & \varphi_4^{(2)} & \psi_4 & \psi_4^{(1)} & 0 & 0 & 0 \\ \phi_1 & \phi_1^{(1)} & \phi_1^{(2)} & 0 & 0 & \chi_1 & \chi_1^{(1)} & \chi_1^{(2)} \\ \phi_2 & \phi_2^{(1)} & \phi_2^{(2)} & 0 & 0 & \chi_2 & \chi_2^{(1)} & \chi_2^{(2)} \\ \phi_3 & \phi_3^{(1)} & \phi_3^{(2)} & 0 & 0 & \chi_3 & \chi_3^{(1)} & \chi_3^{(2)} \\ \phi_4 & \phi_4^{(1)} & \phi_4^{(2)} & 0 & 0 & \chi_4 & \chi_4^{(1)} & \chi_4^{(2)} \end{vmatrix}$$

$$\begin{array}{llll} \varphi_1 = \mathrm{e}^{\xi_1}, & \xi_1 = p_1x_1 + p_1^2x_2, & \varphi_2 = \mathrm{e}^{\xi_2}, & \xi_2 = p_2x_1 + p_2^2x_2, \\ \varphi_3 = \mathrm{e}^{-\xi_1^*}, & -\xi_1^* = -p_1^*x_1 + (-p_1^*)^2x_2, & \varphi_4 = \mathrm{e}^{-\xi_2^*}, & -\xi_2^* = -p_2^*x_1 + (-p_2^*)^2x_2, \\ \phi_1 = \mathrm{e}^{\theta_1}, & \theta_1 = s_1x_1 + s_1^2x_2, & \phi_2 = \mathrm{e}^{\theta_2}, & \theta_2 = s_2x_1 + s_2^2x_2, \\ \phi_3 = \mathrm{e}^{-\theta_1^*}, & -\theta_1^* = -s_1^*x_1 + (-s_1^*)^2x_2, & \phi_4 = \mathrm{e}^{-\theta_2^*}, & -\theta_2^* = -s_2^*x_1 + (-s_2^*)^2x_2, \\ \psi_1 = a_1\,\mathrm{e}^{\eta_1}, & \eta_1 = q_1y_1 + \eta_{10}, & \psi_2 = a_2\,\mathrm{e}^{\eta_2}, & \eta_2 = q_2y_1 + \eta_{20}, \\ \psi_3 = a_3\,\mathrm{e}^{-\eta_1^*}, & -\eta_1^* = -q_1^*y_1 - \eta_{10}^*, & \psi_4 = a_4\,\mathrm{e}^{-\eta_2^*}, & -\eta_2^* = -q_2^*y_1 - \eta_{20}^*, \\ \chi_1 = b_1\,\mathrm{e}^{\zeta_1}, & \zeta_1 = r_1z_1 + \zeta_{10}, & \chi_2 = b_2\,\mathrm{e}^{\zeta_2}, & \zeta_2 = r_2z_1 + \zeta_{20}, \\ \chi_3 = b_3\,\mathrm{e}^{-\zeta_1^*}, & -\zeta_1^* = -r_1^*z_1 - \zeta_{10}^*, & \chi_4 = b_4\,\mathrm{e}^{-\zeta_2^*}, & -\zeta_2^* = -r_2^*z_1 - \zeta_{20}^*, \end{array}$$

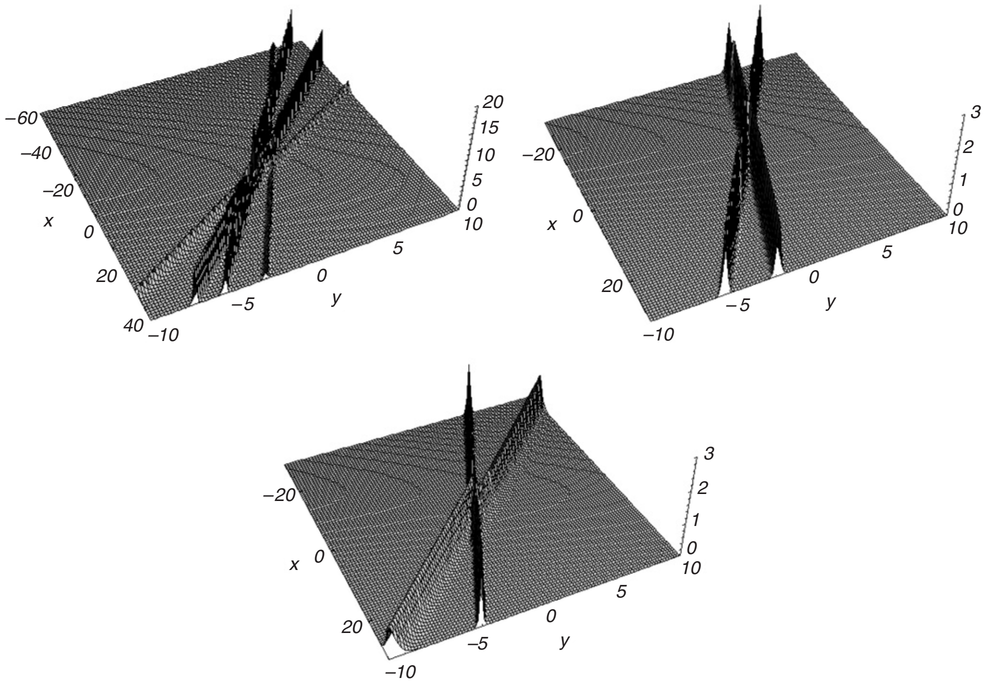

Figure 2. (2, 2, 4)-soliton solution. $p_1 = 2 + 2\mathrm{i}$, $p_2 = 1 + 3\mathrm{i}$, $p_3 = 2 - 2\mathrm{i}$, $p_4 = 1 - 3\mathrm{i}$, $s_1 = -1 + \mathrm{i}$, $s_2 = -2 + 3\mathrm{i}$, $s_3 = -1 - \mathrm{i}$, $s_4 = -2 - 3\mathrm{i}$, $q_1 = -2 + \mathrm{i}$, $q_2 = -3 + 2\mathrm{i}$, $q_3 = -2 - \mathrm{i}$, $q_4 = -3 - 2\mathrm{i}$, $r_1 = 1 + \mathrm{i}$, $r_2 = 1.5 + \mathrm{i}$, $r_3 = 1 - \mathrm{i}$, $r_4 = 1.5 - \mathrm{i}$, $\varepsilon_1 = \varepsilon_2 = \delta_1 = \delta_2 = 1$. The top left graph is $-L$, the top right graph is $S^{(1)}$ and the bottom graph is $S^{(2)}$ at $t = 3$.

$$a_1 = \frac{p_2 - p_1}{q_2 - q_1}(s_1 - p_1)(s_2 - p_1), \qquad b_1 = \frac{s_2 - s_1}{r_2 - r_1}(p_1 - s_1)(p_2 - s_1),$$
$$a_2 = \frac{p_1 - p_2}{q_1 - q_2}(s_1 - p_2)(s_2 - p_2), \qquad b_2 = \frac{s_1 - s_2}{r_1 - r_2}(p_1 - s_2)(p_2 - s_2),$$
$$a_3 = \varepsilon_1 \frac{p_1 + p_1^*}{q_1 + q_1^*}\frac{p_2 + p_1^*}{q_2 + q_1^*}(s_1 + p_1^*)(s_2 + p_1^*), \qquad b_3 = \delta_1 \frac{s_1 + s_1^*}{r_1 + r_1^*}\frac{s_2 + s_1^*}{r_2 + r_1^*}(s_1^* + p_1)(s_1^* + p_2),$$
$$a_4 = \varepsilon_2 \frac{p_1 + p_2^*}{q_1 + q_2^*}\frac{p_2 + p_2^*}{q_2 + q_2^*}(s_1 + p_2^*)(s_2 + p_2^*), \qquad b_4 = \delta_2 \frac{s_1 + s_2^*}{r_1 + r_2^*}\frac{s_2 + s_2^*}{r_2 + r_2^*}(s_2^* + p_1)(s_2^* + p_2),$$
$$\varepsilon_i = \pm 1, \qquad \delta_i = \pm 1 \quad (i = 1, 2),$$

and $x_1 = x$, $x_2 = -\mathrm{i}y$, $y_1 = y - t$, $z_1 = y - t$.

Figures 2 and 3 show the (2, 2, 4)-soliton interaction. In figure 3, an interesting soliton interaction of V-shape solitons and breather-type solitons is found. This interesting interaction is made by the effect of the complicated condition for reality and complex conjugacy. The similar interaction patterns were found in the case of one component [15].

4. Concluding remarks

We have derived two-component analogue of the two-dimensional long wave–short wave interaction equations. Then, we have presented Wronskian solutions to the system. Interestingly, the direction of a soliton on $S^{(1)}$ is different from one of a solitons on $S^{(2)}$.

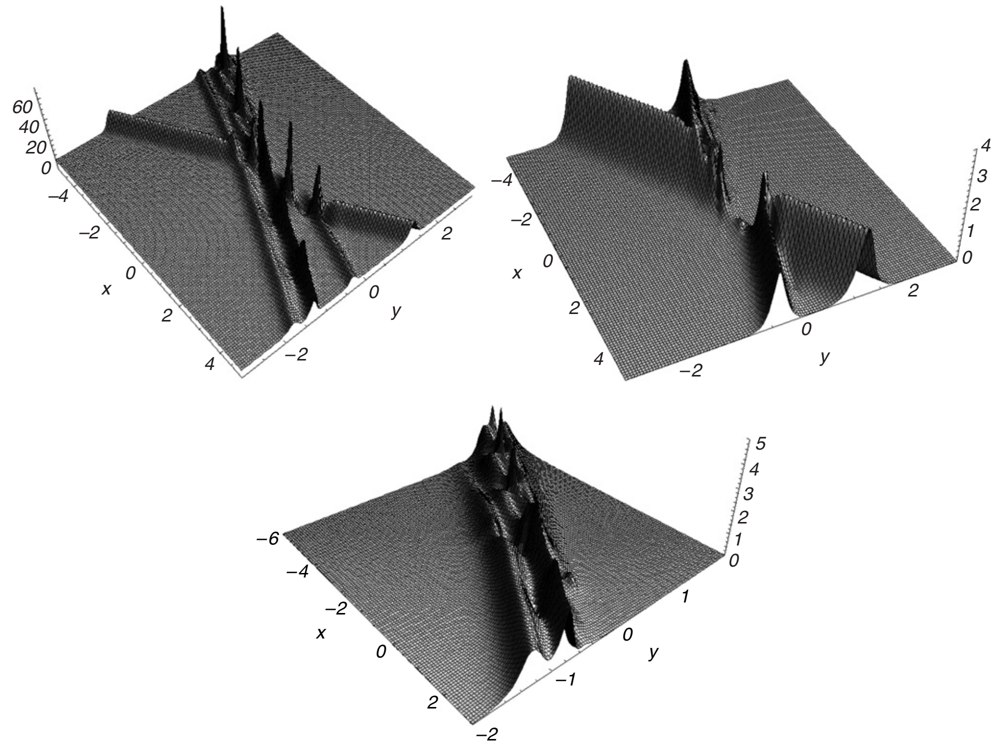

Figure 3. (2, 2, 4)-soliton solution. $p_1 = 2 + 3\mathrm{i}$, $p_2 = 3 - \mathrm{i}$, $p_3 = 2 - 3\mathrm{i}$, $p_4 = 3 + \mathrm{i}$, $s_1 = 2 + 2\mathrm{i}$, $s_2 = 4 + 2\mathrm{i}$, $s_3 = 2 - 2\mathrm{i}$, $s_4 = -4 - 2\mathrm{i}$, $q_1 = 2 + \mathrm{i}$, $q_2 = 2.01 + \mathrm{i}$, $q_3 = 2 - \mathrm{i}$, $q_4 = 2.01 - \mathrm{i}$, $r_1 = 1 + \mathrm{i}$, $r_2 = 1.5 + \mathrm{i}$, $r_3 = 1 - \mathrm{i}$, $r_4 = 1.5 - \mathrm{i}$, $\varepsilon_1 = \varepsilon_2 = \delta_1 = \delta_2 = -1$. The top left graph is $-L$, the top right graph is $S^{(1)}$ and the bottom graph is $S^{(2)}$ at $t = 0$.

This is a special phenomenon in the case of two-dimensional vector soliton. We also found the interesting soliton interaction patterns including V-shape and breather-type soliton solutions.

Note that the generalization to an N-component system is possible. In the N-component case, the solutions are constructed from $(N + 1)$-component Wronskian of the $(N + 1)$-component KP hierarchy.

Recently, another two-dimensional analogue of long wave–short wave interaction equations was derived asymptotically in [25]. The integrability and the existence of N-soliton solution of this system is unknown. The study of multi-component generalization of this system is also interesting.

The detail of analysis of soliton interaction will be presented in the forthcoming paper.

Acknowledgments

We are grateful to Dr Takayuki Tsuchida for many helpful discussions. K M acknowledges support from Faculty Research Council at the University of Texas-Pan American.

Appendix

We show how to derive the elements of Wronskian satisfying the condition

$$\left(\frac{g}{f}\right)^* = \frac{\bar{g}}{f}, \qquad \left(\frac{h}{f}\right)^* = \frac{\bar{h}}{f}, \qquad f\mathcal{G} : \text{real}.$$

Consider the following three-component Wronskian:

$$\tau_{NML} = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \cdots & \varphi_1^{(N-1)} & \vdots & \psi_1 & \psi_1^{(1)} & \cdots & \psi_1^{(M-1)} & \vdots & \chi_1 & \chi_1^{(1)} & \cdots & \chi_1^{(L-1)} \\ \varphi_2 & \varphi_2^{(1)} & \cdots & \varphi_2^{(N-1)} & \vdots & \psi_2 & \psi_2^{(1)} & \cdots & \psi_2^{(M-1)} & \vdots & \chi_2 & \chi_2^{(1)} & \cdots & \chi_2^{(L-1)} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \varphi_{N+M+L} & \varphi_{N+M+L}^{(1)} & \cdots & \varphi_{N+M+L}^{(N-1)} & \vdots & \psi_{N+M+L} & \psi_{N+M+L}^{(1)} & \cdots & \psi_{N+M+L}^{(M-1)} & \vdots & \chi_{N+M+L} & \chi_{N+M+L}^{(1)} & \cdots & \chi_{N+M+L}^{(L-1)} \end{vmatrix},$$

where φ_i are functions of x_1, x_2 and satisfy

$$\partial_{x_2}\varphi_i = \partial_{x_1}^2\varphi_i,$$

ψ_i are arbitrary functions of y_1, χ_i are arbitrary functions of z_1, and

$$\varphi_i^{(n)} = \partial_{x_1}^n\varphi_i, \qquad \psi_i^{(n)} = \partial_{y_1}^n\psi_i, \qquad \chi_i^{(n)} = \partial_{z_1}^n\chi_i.$$

This τ_{NML} satisfies

$$\begin{aligned} &\left(D_{x_1}^2 - D_{x_2}\right)\tau_{N+1,M-1,L}\cdot\tau_{NML} = 0,\\ &\left(D_{x_1}^2 - D_{x_2}\right)\tau_{N+1,M,L-1}\cdot\tau_{NML} = 0,\\ &D_{x_1}D_{y_1}\tau_{NML}\cdot\tau_{NML} = 2\tau_{N+1,M-1,L}\tau_{N-1,M+1,L},\\ &D_{x_1}D_{z_1}\tau_{NML}\cdot\tau_{NML} = 2\tau_{N+1,M,L-1}\tau_{N-1,M,L+1}. \end{aligned}$$

Setting

$$\begin{aligned} &f = \tau_{NML}, && g = \tau_{N+1,M-1,L}, && \bar{g} = \tau_{N-1,M+1,L},\\ &h = \tau_{N+1,M,L-1}, && \bar{h} = \tau_{N-1,M,L+1}, \end{aligned}$$

these τ-functions satisfy

$$\begin{aligned} &\left(D_{x_1}^2 - D_{x_2}\right)g\cdot f = 0,\\ &\left(D_{x_1}^2 + D_{x_2}\right)\bar{g}\cdot f = 0,\\ &\left(D_{x_1}^2 - D_{x_2}\right)h\cdot f = 0,\\ &\left(D_{x_1}^2 + D_{x_2}\right)\bar{h}\cdot f = 0,\\ &D_{x_1}\left(D_{y_1} + D_{z_1}\right)f\cdot f = 2(g\bar{g} + h\bar{h}). \end{aligned}$$

Applying the transformation of the dependent variables

$$S_1 = \frac{g}{f}, \qquad \bar{S}_1 = \frac{\bar{g}}{f}, \qquad S_2 = \frac{h}{f}, \qquad \bar{S}_2 = \frac{\bar{h}}{f}, \qquad L = -(2\log f)_{x_1x_1},$$

we obtain

$$\begin{aligned} &\partial_{x_1}^2 S_1 - LS_1 - \partial_{x_2}S_1 = 0,\\ &\partial_{x_1}^2 \bar{S}_1 - L\bar{S}_1 + \partial_{x_2}\bar{S}_1 = 0,\\ &\partial_{x_1}^2 S_2 - LS_2 - \partial_{x_2}S_2 = 0,\\ &\partial_{x_1}^2 \bar{S}_2 - L\bar{S}_2 + \partial_{x_2}\bar{S}_2 = 0,\\ &-\left(\partial_{y_1} + \partial_{z_1}\right)L = 2(S_1\bar{S}_1 + S_2\bar{S}_2)_{x_1}. \end{aligned}$$

Applying the change of independent variables

$$x_1 = x, \qquad x_2 = -\mathrm{i}y, \qquad y_1 = y - t, \qquad z_1 = y - t,$$

i.e.,

$$\partial_x = \partial_{x_1}, \qquad \partial_y = \partial_{y_1} + \partial_{z_1} - \mathrm{i}\partial_{x_2}, \qquad \partial_t = -\partial_{y_1} - \partial_{z_1},$$

we obtain

$$\begin{aligned}
&\partial_x^2 S_1 - LS_1 - \mathrm{i}(\partial_t + \partial_y)S_1 = 0, \qquad && \partial_x^2 \bar{S}_1 - L\bar{S}_1 + \mathrm{i}(\partial_t + \partial_y)\bar{S}_1 = 0,\\
&\partial_x^2 S_2 - LS_2 - \mathrm{i}(\partial_t + \partial_y)S_2 = 0, && \partial_x^2 \bar{S}_2 - L\bar{S}_2 + \mathrm{i}(\partial_t + \partial_y)\bar{S}_2 = 0,\\
&L_t = 2(S_1\bar{S}_1 + S_2\bar{S}_2)_x.
\end{aligned}$$

In the above solution, we consider the replacements of N, M and L by

$$N \to 2N, \qquad M \to N, \qquad L \to N,$$

i.e., consider

$$\begin{aligned}
&f = \tau_{2N,N,N} \qquad && g = \tau_{2N+1,N-1,N} \qquad && \bar{g} = \tau_{2N-1,N+1,N}\\
&h = \tau_{2N+1,N,N-1} && \bar{h} = \tau_{2N-1,N,N+1}.
\end{aligned}$$

Setting

$$\psi_{2N+1} = \psi_{2N+2} = \cdots = \psi_{4N} = 0, \qquad \chi_1 = \chi_2 = \cdots = \chi_{2N} = 0,$$

we have

$$\tau_{2N+n+m,N-n,N-m} = \left|\begin{array}{cccc|cccc|cccc}
\varphi_1 & \varphi_1^{(1)} & \cdots & \varphi_1^{(2N+n+m-1)} & \psi_1 & \psi_1^{(1)} & \cdots & \psi_1^{(N-n-1)} & & & & \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & & \mathbf{0} & \\
\varphi_{2N} & \varphi_{2N}^{(1)} & \cdots & \varphi_{2N}^{(2N+n+m-1)} & \psi_{2N} & \psi_{2N}^{(1)} & \cdots & \psi_{2N}^{(N-n-1)} & & & & \\
\cdots & & & & \cdots & & & & \cdots & & & \\
\varphi_{2N+1} & \varphi_{2N+1}^{(1)} & \cdots & \varphi_{2N+1}^{(2N+n+m-1)} & & & & & \chi_{2N+1} & \chi_{2N+1}^{(1)} & \cdots & \chi_{2N+1}^{(N-m-1)} \\
\vdots & \vdots & & \vdots & & & \mathbf{0} & & \vdots & \vdots & & \vdots \\
\varphi_{4N} & \varphi_{4N}^{(1)} & \cdots & \varphi_{4N}^{(2N+n+m-1)} & & & & & \chi_{4N} & \chi_{4N}^{(1)} & \cdots & \chi_{4N}^{(N-m-1)}
\end{array}\right|.$$

Note that this determinant is 0 if $N - n > 2N$ or $N - m > 2N$. Consider the following table of nonzero τ-functions:

$$\begin{array}{cccccc}
m = N & \tau_{2N,2N,0} & \tau_{2N+1,2N-1,0} & \cdots & \tau_{4N-1,1,0} & \tau_{4N,0,0}\\
m = N-1 & \tau_{2N-1,2N,1} & \tau_{2N,2N-1,1} & \cdots & \tau_{4N-2,1,1} & \tau_{4N-1,0,1}\\
 & \vdots & \vdots & & \vdots & \vdots\\
m = -N+1 & \tau_{1,2N,2N-1} & \tau_{2,2N-1,2N-1} & \cdots & \tau_{2N,1,2N-1} & \tau_{2N+1,0,2N-1}\\
m = -N & \tau_{0,2N,2N} & \tau_{1,2N-1,2N} & \cdots & \tau_{2N-1,1,2N} & \tau_{2N,0,2N}\\
 & n = -N & n = -N+1 & & n = N-1 & n = N
\end{array}$$

The τ-function in the centre on the table is corresponding to f:

$$\begin{array}{ccccccc}
\tau_{2N,2N,0} & \tau_{2N+1,2N-1,0} & \cdots & \cdots & \cdots & \tau_{4N-1,1,0} & \tau_{4N,0,0}\\
\tau_{2N-1,2N,1} & \tau_{2N,2N-1,1} & \cdots & \cdots & \cdots & \tau_{4N-2,1,1} & \tau_{4N-1,0,1}\\
\vdots & \vdots & & h & & \vdots & \vdots\\
\vdots & \vdots & \bar{g} & f & g & \vdots & \vdots\\
\vdots & \vdots & & \bar{h} & & \vdots & \vdots\\
\tau_{1,2N,2N-1} & \tau_{2,2N-1,2N-1} & \cdots & \cdots & \cdots & \tau_{2N,1,2N-1} & \tau_{2N+1,0,2N-1}\\
\tau_{0,2N,2N} & \tau_{1,2N-1,2N} & \cdots & \cdots & \cdots & \tau_{2N-1,1,2N} & \tau_{2N,0,2N}.
\end{array}$$

Now we want to find the condition of complex conjugacy (f: real, $\bar{g} = g^*$, $\bar{h} = h^*$, where * means complex conjugate). For the function $\tau_{2N+n+m,N-n,N-m}$, the bilinear equations of two-dimensional Toda lattice

$$D_{x_1} D_{y_1} \tau_{2N+n+m,N-n,N-m} \cdot \tau_{2N+n+m,N-n,N-m} = 2\tau_{2N+n+m+1,N-n-1,N-m} \tau_{2N+n+m-1,N-n+1,N-m},$$
$$D_{x_1} D_{z_1} \tau_{2N+n+m,N-n,N-m} \cdot \tau_{2N+n+m,N-n,N-m} = 2\tau_{2N+n+m+1,N-n,N-m-1} \tau_{2N+n+m-1,N-n,N-m+1},$$

are satisfied. Since the function $\tau_{4N,0,0}$ depends on only x_1, x_2, the following bilinear equations are satisfied:

$$D_{x_1} D_{y_1} \frac{\tau_{2N+n+m,N-n,N-m}}{\tau_{4N,0,0}} \cdot \frac{\tau_{2N+n+m,N-n,N-m}}{\tau_{4N,0,0}} = 2 \frac{\tau_{2N+n+m+1,N-n-1,N-m}}{\tau_{4N,0,0}} \frac{\tau_{2N+n+m-1,N-n+1,N-m}}{\tau_{4N,0,0}},$$
$$D_{x_1} D_{z_1} \frac{\tau_{2N+n+m,N-n,N-m}}{\tau_{4N,0,0}} \cdot \frac{\tau_{2N+n+m,N-n,N-m}}{\tau_{4N,0,0}} = 2 \frac{\tau_{2N+n+m+1,N-n,N-m-1}}{\tau_{4N,0,0}} \frac{\tau_{2N+n+m-1,N-n,N-m+1}}{\tau_{4N,0,0}}.$$

The table of solutions of these bilinear equations is as follows:

$$\begin{array}{ccccccc}
\frac{\tau_{2N,2N,0}}{\tau_{4N,0,0}} & \frac{\tau_{2N+1,2N-1,0}}{\tau_{4N,0,0}} & \cdots & \cdots & \cdots & \frac{\tau_{4N-1,1,0}}{\tau_{4N,0,0}} & 1 \\
\frac{\tau_{2N-1,2N,1}}{\tau_{4N,0,0}} & \frac{\tau_{2N,2N-1,1}}{\tau_{4N,0,0}} & \cdots & \cdots & \cdots & \frac{\tau_{4N-2,1,1}}{\tau_{4N,0,0}} & \frac{\tau_{4N-1,0,1}}{\tau_{4N,0,0}} \\
\vdots & \vdots & & \frac{h}{\tau_{4N,0,0}} & & \vdots & \vdots \\
\vdots & \vdots & \frac{\bar{g}}{\tau_{4N,0,0}} & \frac{f}{\tau_{4N,0,0}} & \frac{g}{\tau_{4N,0,0}} & \vdots & \vdots \\
\vdots & \vdots & & \frac{\bar{h}}{\tau_{4N,0,0}} & & \vdots & \vdots \\
\frac{\tau_{1,2N,2N-1}}{\tau_{4N,0,0}} & \frac{\tau_{2,2N-1,2N-1}}{\tau_{4N,0,0}} & \cdots & \cdots & \cdots & \frac{\tau_{2N,1,2N-1}}{\tau_{4N,0,0}} & \frac{\tau_{2N+1,0,2N-1}}{\tau_{4N,0,0}} \\
\frac{\tau_{0,2N,2N}}{\tau_{4N,0,0}} & \frac{\tau_{1,2N-1,2N}}{\tau_{4N,0,0}} & \cdots & \cdots & \cdots & \frac{\tau_{2N-1,1,2N}}{\tau_{4N,0,0}} & \frac{\tau_{2N,0,2N}}{\tau_{4N,0,0}}.
\end{array}$$

Since the function $\tau_{0,2N,2N}$ does not depend on x_1, the following bilinear equations are also satisfied:

$$D_{x_1} D_{y_1} \frac{\tau_{2N+n+m,N-n,N-m}}{\tau_{0,2N,2N}} \cdot \frac{\tau_{2N+n+m,N-n,N-m}}{\tau_{0,2N,2N}} = 2 \frac{\tau_{2N+n+m+1,N-n-1,N-m}}{\tau_{0,2N,2N}} \frac{\tau_{2N+n+m-1,N-n+1,N-m}}{\tau_{0,2N,2N}},$$
$$D_{x_1} D_{z_1} \frac{\tau_{2N+n+m,N-n,N-m}}{\tau_{0,2N,2N}} \cdot \frac{\tau_{2N+n+m,N-n,N-m}}{\tau_{0,2N,2N}} = 2 \frac{\tau_{2N+n+m+1,N-n,N-m-1}}{\tau_{0,2N,2N}} \frac{\tau_{2N+n+m-1,N-n,N-m+1}}{\tau_{0,2N,2N}}.$$

The table of solutions of these bilinear equations is as follows:

$$\begin{array}{ccccccc}
\frac{\tau_{2N,2N,0}}{\tau_{0,2N,2N}} & \frac{\tau_{2N+1,2N-1,0}}{\tau_{0,2N,2N}} & \cdots & \cdots & \cdots & \frac{\tau_{4N-1,1,0}}{\tau_{0,2N,2N}} & \frac{\tau_{4N,0,0}}{\tau_{0,2N,2N}} \\
\frac{\tau_{2N-1,2N,1}}{\tau_{0,2N,2N}} & \frac{\tau_{2N,2N-1,1}}{\tau_{0,2N,2N}} & \cdots & \cdots & \cdots & \frac{\tau_{4N-2,1,1}}{\tau_{0,2N,2N}} & \frac{\tau_{4N-1,0,1}}{\tau_{0,2N,2N}} \\
\vdots & \vdots & & \frac{h}{\tau_{0,2N,2N}} & & \vdots & \vdots \\
\vdots & \vdots & \frac{\bar{g}}{\tau_{0,2N,2N}} & \frac{f}{\tau_{0,2N,2N}} & \frac{g}{\tau_{0,2N,2N}} & \vdots & \vdots \\
\vdots & \vdots & & \frac{\bar{h}}{\tau_{0,2N,2N}} & & \vdots & \vdots \\
\frac{\tau_{1,2N,2N-1}}{\tau_{0,2N,2N}} & \frac{\tau_{2,2N-1,2N-1}}{\tau_{0,2N,2N}} & \cdots & \cdots & \cdots & \frac{\tau_{2N,1,2N-1}}{\tau_{0,2N,2N}} & \frac{\tau_{2N+1,0,2N-1}}{\tau_{0,2N,2N}} \\
1 & \frac{\tau_{1,2N-1,2N}}{\tau_{0,2N,2N}} & \cdots & \cdots & \cdots & \frac{\tau_{2N-1,1,2N}}{\tau_{0,2N,2N}} & \frac{\tau_{2N,0,2N}}{\tau_{0,2N,2N}}.
\end{array}$$

Consider the case in which the relations

$$\frac{\tau_{4N-1,1,0}}{\tau_{4N,0,0}} = \left(\frac{\tau_{1,2N-1,2N}}{\tau_{0,2N,2N}}\right)^*,$$

and

$$\frac{\tau_{4N-1,0,1}}{\tau_{4N,0,0}} = \left(\frac{\tau_{1,2N,2N-1}}{\tau_{0,2N,2N}}\right)^*,$$

are satisfied. If we choose elements in the solution in section 3 as $\varphi_i, \psi_i, \chi_i$, these relations are satisfied, and we also note that the relations

$$\frac{f}{\tau_{4N,0,0}} = \left(\frac{f}{\tau_{0,2N,2N}}\right)^*,$$

$$\frac{g}{\tau_{4N,0,0}} = \left(\frac{\bar{g}}{\tau_{0,2N,2N}}\right)^*,$$

$$\frac{h}{\tau_{4N,0,0}} = \left(\frac{\bar{h}}{\tau_{0,2N,2N}}\right)^*,$$

are satisfied. From these relations, we conclude that the complex conjugacy condition

$$\left(\frac{g}{f}\right)^* = \frac{\bar{g}}{f}, \qquad \left(\frac{h}{f}\right)^* = \frac{\bar{h}}{f}, \qquad f\mathcal{G} : \text{real},$$

where $\mathcal{G}$ is an exponential factor which is a gauge function, is satisfied for the solution presented in section 3.

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